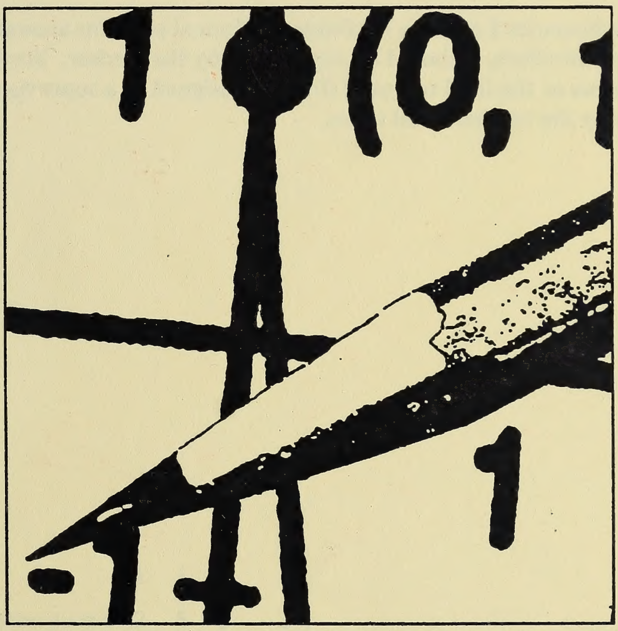
MATHEMATICS 31

LEARNING FACILITATOR'S MANUAL



UNIT 9: MATRICES AND LINEAR TRANSFORMATIONS

Distance Learning

Alberta EDUCATION

Note

This Mathematics Learning Facilitator's Manual contains answers to teacher-assessed assignments and the final test; therefore, it should be kept secure by the teacher. Students should not have access to these assignments or the final test until they are assigned in a supervised situation. The answers should be stored securely by the teacher at all times.

Mathematics 31
Learning Facilitator's Manual
Unit 9
Matrices and Linear Transformations
Alberta Distance Learning Centre
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Topic 1: Operations Defined on Matrices

2 1. State the dimensions of each of the following matrices.

a. $\begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix}$

The dimensions are 3×1.

b. $\begin{bmatrix} 0 & 2 & 3 & 5 \\ 1 & 4 & 8 & 7 \end{bmatrix}$

The dimensions are 2×4.

3 2. Answer the following given the matrix $\begin{bmatrix} 1 & 3 & 0 & 7 \\ 2 & 5 & 8 & -2 \\ 3 & 9 & 0 & 4 \end{bmatrix}$.

a. Identify a_{23}.

$a_{23} = 8$

b. Find $a_{11} + a_{32}$.

$$\begin{aligned} a_{11} + a_{32} &= 1 + 9 \\ &= 10 \end{aligned}$$

③ 3. Find $A+B$ if $A=\begin{bmatrix} 0 & 3 & 4 \\ 2 & 6 & 1 \\ -3 & 5 & -2 \end{bmatrix}$ and $B=\begin{bmatrix} -1 & 3 & 1 \\ 0 & 2 & -3 \\ 4 & -2 & 5 \end{bmatrix}$.

$$A+B=\begin{bmatrix} 0-1 & 3+3 & 4+1 \\ 2+0 & 6+2 & 1-3 \\ -3+4 & 5-2 & -2+5 \end{bmatrix}$$
$$=\begin{bmatrix} -1 & 6 & 5 \\ 2 & 8 & -2 \\ 1 & 3 & 3 \end{bmatrix}$$

③ 4. Find $-3A$ if $A=\begin{bmatrix} 2 & -3 \\ 5 & 1 \end{bmatrix}$.

$$-3A=\begin{bmatrix} (-3)(2) & (-3)(-3) \\ (-3)(5) & (-3)(1) \end{bmatrix}$$
$$=\begin{bmatrix} -6 & 9 \\ -15 & -3 \end{bmatrix}$$

4

5. Find $A-2B$ if $A=\begin{bmatrix} 3 & 4 \\ 8 & 7 \end{bmatrix}$ and $B=\begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}$.

$$
\begin{aligned}
A-2B &= \begin{bmatrix} 3 & 4 \\ 8 & 7 \end{bmatrix} - \begin{bmatrix} (2)(1) & (2)(-2) \\ (2)(0) & (2)(3) \end{bmatrix} \\
&= \begin{bmatrix} 3 & 4 \\ 8 & 7 \end{bmatrix} - \begin{bmatrix} 2 & -4 \\ 0 & 6 \end{bmatrix} \\
&= \begin{bmatrix} 3-2 & 4+4 \\ 8-0 & 7-6 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 8 \\ 8 & 1 \end{bmatrix}
\end{aligned}
$$

4 6. Find AB if $A=\begin{bmatrix}3 & 1 & -2\\ 0 & 5 & -3\end{bmatrix}$ and $B=\begin{bmatrix}1 & 5\\ 2 & 0\\ 4 & -3\end{bmatrix}$.

$$\begin{aligned}AB &= \begin{bmatrix}3 & 1 & -2\\ 0 & 5 & -3\end{bmatrix}\begin{bmatrix}1 & 5\\ 2 & 0\\ 4 & -3\end{bmatrix}\\ &= \begin{bmatrix}(3)(1)+(1)(2)+(-2)(4) & (3)(5)+(1)(0)+(-2)(-3)\\ 0(1)+(5)(2)+(-3)(4) & 0(5)+(5)(0)+(-3)(-3)\end{bmatrix}\\ &= \begin{bmatrix}-3 & 21\\ -2 & 9\end{bmatrix}\end{aligned}$$

5 7. Find A^2-2B if $A=\begin{bmatrix}1 & -2\\ -3 & 4\end{bmatrix}$ and $B=\begin{bmatrix}3 & 0\\ 4 & 1\end{bmatrix}$.

$$\begin{aligned}A^2 &= \begin{bmatrix}1 & -2\\ -3 & 4\end{bmatrix}\begin{bmatrix}1 & -2\\ -3 & 4\end{bmatrix}\\ &= \begin{bmatrix}7 & -10\\ -15 & 22\end{bmatrix}\end{aligned}$$

$$2B=\begin{bmatrix}6 & 0\\ 8 & 2\end{bmatrix}$$

$$\begin{aligned}\therefore A^2-2B &= \begin{bmatrix}7 & -10\\ -15 & 22\end{bmatrix}-\begin{bmatrix}6 & 0\\ 8 & 2\end{bmatrix}\\ &= \begin{bmatrix}1 & -10\\ -23 & 20\end{bmatrix}\end{aligned}$$

6

8. Show that $A(B+C)=AB+AC$ if $A=\begin{bmatrix}-2 & 1\\ 2 & 3\end{bmatrix}$, $B=\begin{bmatrix}4 & -1\\ 0 & -2\end{bmatrix}$, and $C=\begin{bmatrix}-3 & 2\\ -2 & 4\end{bmatrix}$.

$$\begin{aligned}A(B+C)&=\begin{bmatrix}-2 & 1\\ 2 & 3\end{bmatrix}\left(\begin{bmatrix}4 & -1\\ 0 & -2\end{bmatrix}+\begin{bmatrix}-3 & 2\\ -2 & 4\end{bmatrix}\right)\\ &=\begin{bmatrix}-2 & 1\\ 2 & 3\end{bmatrix}\begin{bmatrix}1 & 1\\ -2 & 2\end{bmatrix}\\ &=\begin{bmatrix}-2-2 & -2+2\\ 2-6 & 2+6\end{bmatrix}\\ &=\begin{bmatrix}-4 & 0\\ -4 & 8\end{bmatrix}\end{aligned}$$

$$\begin{aligned}AB+AC&=\begin{bmatrix}-2 & 1\\ 2 & 3\end{bmatrix}\begin{bmatrix}4 & -1\\ 0 & -2\end{bmatrix}+\begin{bmatrix}-2 & 1\\ 2 & 3\end{bmatrix}\begin{bmatrix}-3 & 2\\ -2 & 4\end{bmatrix}\\ &=\begin{bmatrix}-8+0 & 2-2\\ 8+0 & -2-6\end{bmatrix}+\begin{bmatrix}6-2 & -4+4\\ -6-6 & 4+12\end{bmatrix}\\ &=\begin{bmatrix}-8 & 0\\ 8 & -8\end{bmatrix}+\begin{bmatrix}4 & 0\\ -12 & 16\end{bmatrix}\\ &=\begin{bmatrix}-4 & 0\\ -4 & 8\end{bmatrix}\end{aligned}$$

Therefore, $A(B+C)=AB+AC$.

Topic 1

_______ marks

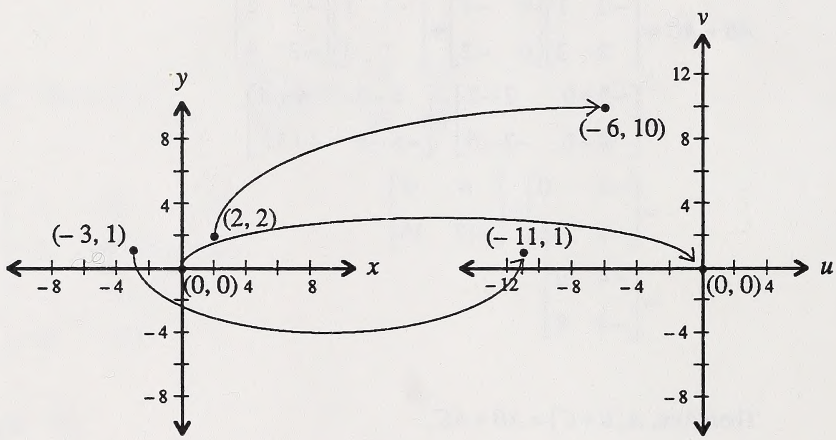
Topic 2: Linear Transformation and Its Inverse

8 1. Under the transformation $T\begin{cases} u = 2x - 5y \\ v = x + 4y \end{cases}$, find the images of the points $(0, 0)$, $(2, 2)$, and $(-3, 1)$. Show the points and their images on a diagram.

$T\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ The image of $(0, 0)$ is $(0, 0)$.

$T\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -6 \\ 10 \end{bmatrix}$ The image of $(2, 2)$ is $(-6, 10)$.

$T\begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} -11 \\ 1 \end{bmatrix}$ The image of $(-3, 1)$ is $(-11, 1)$.



4

2. Determine whether the following mappings are onto or into mappings. Use either the word *into* or *onto* to complete the statements.

   a. $M\begin{cases} u = 2x \\ v = 3y \end{cases}$ is a transformation of the xy-plane ___**onto**___ the uv-plane.

   b. $N\begin{cases} r = x - 3y \\ s = 5x - 15y \end{cases}$ is a transformation of the xy-plane ___**into**___ the rs-plane, or it is a transformation of the xy-plane ___**onto**___ the line $s = 5r$.

6

3. Find (u, v) under $T\begin{cases} m = 2u - 3v \\ n = -u + v \end{cases}$ if $T\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

$$3 = 2u - 3v \quad (1)$$
$$-2 = -u + v \quad (2)$$

$(1) + 2 \times (2)$: $-1 = -v$

$$v = 1$$

$(1) + 3 \times (2)$: $-3 = -u$

$$u = 3$$

Therefore, $(u, v) = (3, 1)$.

6

4. For the transformation $T\begin{cases} u=2x-5y \\ v=x+3y \end{cases}$, find the point (x, y) if its image is $(-3, 5)$.

$$\begin{aligned} x &= \frac{d}{ad-bc}u - \frac{b}{ad-bc}v \\ &= \frac{3}{11}u - \frac{(-5)}{11}v \\ &= \frac{3}{11}(-3) + \frac{5}{11}(5) \\ &= \frac{16}{11} \end{aligned}$$

$$\begin{aligned} y &= \frac{-c}{ad-bc}u + \frac{a}{ad-bc}v \\ &= \frac{-1}{11}(-3) + \frac{2}{11}(5) \\ &= \frac{13}{11} \end{aligned}$$

Therefore, the point $(x, y) = \left(\frac{16}{11}, \frac{13}{11}\right)$.

6

5. For $T\begin{cases} u=5x+7y \\ v=3x-6y \end{cases}$, find T^{-1} if it exists.

$$\begin{aligned} x &= \frac{d}{ad-bc}u - \frac{b}{ad-bc}v \\ &= \frac{-6}{-51}u - \frac{7}{-51}v \\ &= \frac{6}{51}u + \frac{7}{51}v \end{aligned}$$

$$\begin{aligned} y &= \frac{-c}{ad-bc}u + \frac{a}{ad-bc}v \\ &= \frac{-3}{-51}u + \frac{5}{-51}v \\ &= \frac{3}{51}u - \frac{5}{51}v \end{aligned}$$

Therefore, $T^{-1}\begin{cases} x=\frac{6}{51}u+\frac{7}{51}v \\ y=\frac{3}{51}u-\frac{5}{51}v \end{cases}$ or $T^{-1}\begin{cases} x=\frac{2}{17}u+\frac{7}{51}v \\ y=\frac{1}{17}u-\frac{5}{51}v \end{cases}$.

Topic 2

_______ marks

Topic 3: Product of Two Transformations

(4) 1. Find $WS\begin{bmatrix}2\\5\end{bmatrix}$ if $W\begin{cases}u=3m-5n\\v=4m+n\end{cases}$ and $S\begin{cases}m=2x-3y\\n=5x+y\end{cases}$.

$$\begin{aligned}WS\begin{bmatrix}2\\5\end{bmatrix}&=\begin{bmatrix}3&-5\\4&1\end{bmatrix}\begin{bmatrix}2&-3\\5&1\end{bmatrix}\begin{bmatrix}2\\5\end{bmatrix}\\&=\begin{bmatrix}3&-5\\4&1\end{bmatrix}\begin{bmatrix}-11\\15\end{bmatrix}\\&=\begin{bmatrix}-108\\-29\end{bmatrix}\end{aligned}$$

(6) 2. For the transformations $A\begin{cases}u=2m-n\\v=-m+3n\end{cases}$ and $B\begin{cases}m=-2x+y\\n=x+y\end{cases}$, find $AB\begin{bmatrix}-1\\3\end{bmatrix}$.

$$\begin{aligned}AB\begin{bmatrix}-1\\3\end{bmatrix}&=\begin{bmatrix}2&-1\\-1&3\end{bmatrix}\begin{bmatrix}-2&1\\1&1\end{bmatrix}\begin{bmatrix}-1\\3\end{bmatrix}\\&=\begin{bmatrix}2&-1\\-1&3\end{bmatrix}\begin{bmatrix}5\\2\end{bmatrix}\\&=\begin{bmatrix}2(5)+2(-1)\\5(-1)+2(3)\end{bmatrix}\\&=\begin{bmatrix}8\\1\end{bmatrix}\end{aligned}$$

Topic 3

_______ marks

Topic 4: Special Transformations

4

1. The point $(-2,3)$ is rotated about the origin through 90°. Find the image.

$$R_{\frac{\pi}{2}}\begin{bmatrix}-2\\3\end{bmatrix}=\begin{bmatrix}\cos 90° & -\sin 90°\\ \sin 90° & \cos 90°\end{bmatrix}\begin{bmatrix}-2\\3\end{bmatrix}$$
$$=\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}\begin{bmatrix}-2\\3\end{bmatrix}$$
$$=\begin{bmatrix}-3\\-2\end{bmatrix}$$

Therefore, the image of $(-2,3)$ is $(-3,-2)$.

6

2. $P'Q'$ is the image of the segment determined by $P(2,2)$ and $Q(5,4)$ under the transformation $M=\begin{bmatrix}4 & 0\\0 & 4\end{bmatrix}$. Find the length of $P'Q'$.

$$M\begin{bmatrix}2\\2\end{bmatrix}=\begin{bmatrix}4 & 0\\0 & 4\end{bmatrix}\begin{bmatrix}2\\2\end{bmatrix}$$
$$=\begin{bmatrix}8\\8\end{bmatrix}$$

$$M\begin{bmatrix}5\\4\end{bmatrix}=\begin{bmatrix}4 & 0\\0 & 4\end{bmatrix}\begin{bmatrix}5\\4\end{bmatrix}$$
$$=\begin{bmatrix}20\\16\end{bmatrix}$$

$$P'Q'=\sqrt{(20-8)^2+(16-8)^2}$$
$$=\sqrt{144+64}$$
$$=\sqrt{208}$$
$$=4\sqrt{13}$$

4 3. Segment PQ is determined by $P(3,2)$ and $Q(-2,5)$. PQ is projected on the x-axis. Find the length of the image.

$$P_x\begin{bmatrix}3\\2\end{bmatrix}=\begin{bmatrix}1&0\\0&0\end{bmatrix}\begin{bmatrix}3\\2\end{bmatrix}$$
$$=\begin{bmatrix}3\\0\end{bmatrix}$$

$$P_x\begin{bmatrix}-2\\5\end{bmatrix}=\begin{bmatrix}1&0\\0&0\end{bmatrix}\begin{bmatrix}-2\\5\end{bmatrix}$$
$$=\begin{bmatrix}-2\\0\end{bmatrix}$$

$\therefore$ The length of the image is $3-(-2)=5$.

6 4. Segment PQ is determined by $P(3,-1)$ and $Q(6,-4)$. If PQ is reflected in the x-axis and then in the origin, find the image.

$$R_oR_x=\begin{bmatrix}-1&0\\0&-1\end{bmatrix}\begin{bmatrix}1&0\\0&-1\end{bmatrix}$$
$$=\begin{bmatrix}-1&0\\0&1\end{bmatrix}$$

$$R_oR_x\begin{bmatrix}3\\-1\end{bmatrix}=\begin{bmatrix}-1&0\\0&1\end{bmatrix}\begin{bmatrix}3\\-1\end{bmatrix}$$
$$=\begin{bmatrix}-3\\-1\end{bmatrix}$$

$$R_oR_x\begin{bmatrix}6\\-4\end{bmatrix}=\begin{bmatrix}-1&0\\0&1\end{bmatrix}\begin{bmatrix}6\\-4\end{bmatrix}$$
$$=\begin{bmatrix}-6\\-4\end{bmatrix}$$

The image is segment $P'Q'$ determined by $(-3,-1)$ and $(-6,-4)$.

5

5. Segment AB is determined by $A(-1,2)$ and $B(3,5)$. Find the image of AB under the shear $S_x \begin{cases} u = x + 4y \\ v = y \end{cases}$ in the x-direction.

$$S_x\begin{bmatrix}-1\\2\end{bmatrix}=\begin{bmatrix}1&4\\0&1\end{bmatrix}\begin{bmatrix}-1\\2\end{bmatrix}$$
$$=\begin{bmatrix}7\\2\end{bmatrix}$$

$$S_x\begin{bmatrix}3\\5\end{bmatrix}=\begin{bmatrix}1&4\\0&1\end{bmatrix}\begin{bmatrix}3\\5\end{bmatrix}$$
$$=\begin{bmatrix}23\\5\end{bmatrix}$$

The image is the segment $A'B'$ determined by $A'(7, 2)$ and $B'(23, 5)$.

5

6. Find the matrix that is equivalent to a rotation of 90° followed by a shear in the y-direction. (Constant of proportionality $a = 3$.)

$$S_yR_{\frac{\pi}{2}}=\begin{bmatrix}1&0\\3&1\end{bmatrix}\begin{bmatrix}\cos 90^\circ&-\sin 90^\circ\\\sin 90^\circ&\cos 90^\circ\end{bmatrix}$$
$$=\begin{bmatrix}1&0\\3&1\end{bmatrix}\begin{bmatrix}0&-1\\1&0\end{bmatrix}$$
$$=\begin{bmatrix}0&-1\\1&-3\end{bmatrix}$$

Topic 4

_______ marks

This booklet cannot be purchased separately; the Learning Facilitator's Manual is available only as a complete set.

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